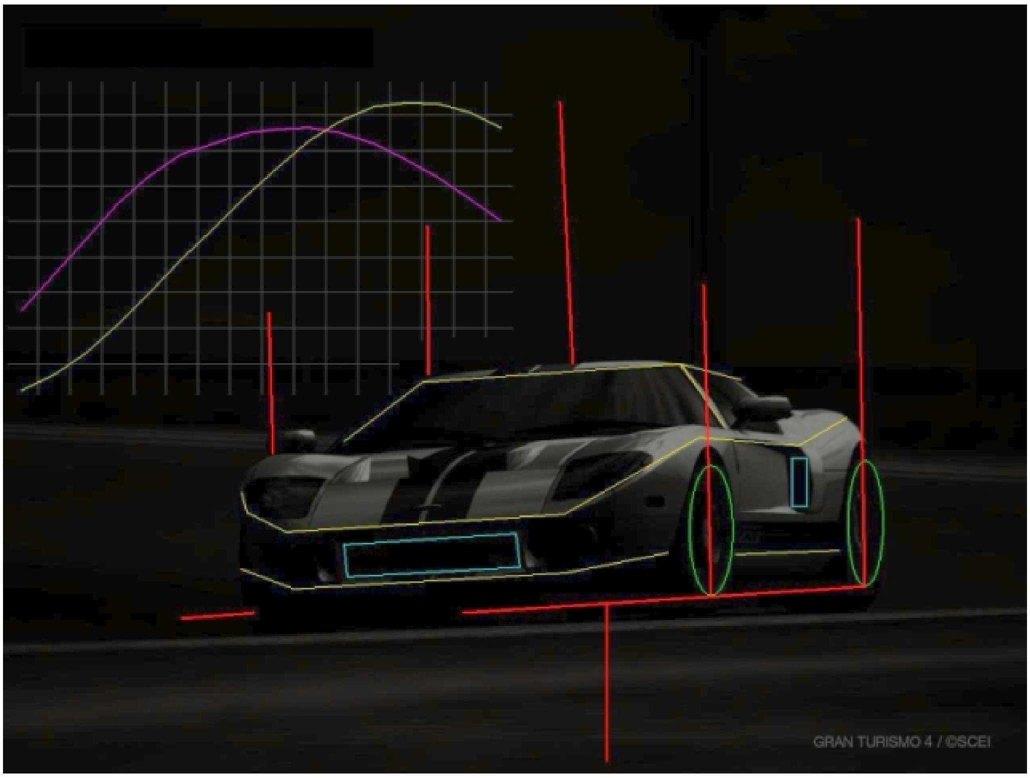
Racing Physics

Applications of Introductory Mechanics to Auto Racing



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Course Description

This course, taught for the MIT Educational Studies Program (ESP) SPLASH! Weekend, 2006, provides one answer to the question, "When is this ever used?" by exploring a few interesting applications of high school-level mechanics to automobile racing. Since the course was designed for two 2-hour sessions, a working knowledge of introductory classical mechanics is assumed. Although relevant concepts and equations are briefly reviewed, the bulk of the course consists of applying concepts to a handful of particular racing examples. The material is split into two parts: Part I deals generally with external forces: linear and centripetal acceleration, friction, gravity, lift (downforce) and drag, and a mass-spring-damper model of suspension. Part II deals with internal forces: engine dynamics, power and torque curves, power transmission and gear ratios, and a look at electric vehicles. Finally, students get a chance to explore racing physics concepts within the very realistic physics engine of Gran Turismo 4, a racing simulator produced by Sony Computer Entertainment, Inc. for the Playstation 2 game console.

About the Authors

Shane Colton is a junior in Mechanical Engineering at MIT. He became interested in engineering through participation on a FIRST Robotics team in high school on Long Island. His interests lie in controls and mechatronics and he recently traveled to Tokyo for the International Design Competition. He is a long-time Gran Turismo fan and achieved 100.0% completion in Gran Turismo 3. (He's still working on Gran Turismo 4.)

Dayán Páez is a Mechanical Engineering junior at MIT. His interests include product design and development, and he has built for the stage, robotics, and even medical devices. He is at home in a wood or metal shop, but also enjoys the frustration of computer programming. A sailor and bongoist, Dayán is at best when engulfed in the aggravating pleasures of nonlinear problems.

1 Physics

Physics provides us with a number of helpful definitions and relations, such as the concept of *displacement*, *velocity*, and *acceleration* which allows us to describe numerically the motion of any mass.

In his Second Law of Motion, Newton defined a force as a 'change in momentum' of a system, and from there we can derive the equation

$$F = ma$$

We would need a textbook to properly treat physics and all the equations. Instead, we will merely present, without proof, a number of equations that we may find useful for this class.

Centripetal Force: a force that holds a mass in circular motion with contant radius, r, and tangential speed, v_t. $F_c = m\frac{v_t^2}{r}$

Friction: a force that resists motion between two surfaces, propotional to the normal force, F_N, between the surfaces and the coefficient a friction, μ (depends on the materials). $F_f = \mu F_N$

Work: a *force* exerted over a distance. $W = F \cdot s$

Power: the rate at which *work* is done. $P = \frac{\Delta W}{\Delta t} = F \cdot v$

Torque: a twisting equivalent to force. A force applied a given distance r from an axis of rotation creates a torque on the system. Think of a tire iron loosening up a lug nut. The force is applied at a distance away from the nut, creating a torque that loosens it. $T = r \times F$

Aerodynamic Forces: Drag and lift are the epitome of non-linear phenomena. For the classic problem of a fluid (air or water, for example) flowing over a surface (like a car or an airplane wing) an agreed-upon equation for the force is

$$F = C\left(\frac{1}{2}\rho v^2 A\right)$$

Where F is the force due to drag (or lift), C is the so-called *coefficient of drag (or lift)*, ρ is the density of the fluid ($1.25\,\mathbf{kg/m^3}$ for air), v is the velocity of the air (or the car), and A is the area exposed to drag or lift. For a vehicle, this area is essentially the frontal area.

The coefficient of drag is not a calculated value. Instead it is determined through experiments. There are a number of tables that provide the coefficient of drag for a number of geometries and known objects.

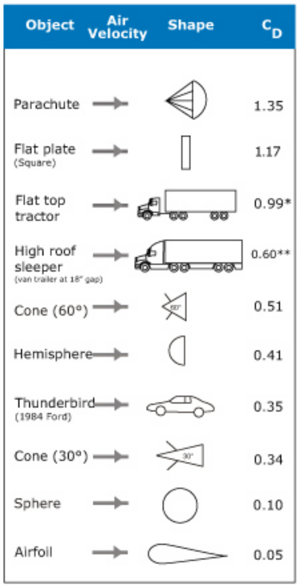

Figure 1: Coefficient of drag for different objects. *(Source: Ford Motor Co. *National Research Council of Canada, **NASA)*

For more information on drag and lift, we recommend the website http://www.aerodyn.org/Drag/.

These equations are useful when analyzing translational motion (i.e. along a line). Rotational systems, though different, are very similar. Table 1 is a summary of the analogous terms in the two domains.

Variable	**Translational**	**Rotational**
Displacement	s	θ
Velocity	$v = s/t$	$\omega = \theta/t$
Acceleration	$a = v/t$	$\alpha = \omega/t$
Force	'Force' $F = ma$	'Torque' $T = I\alpha$
Power	$P = F \cdot v$	$P = T \cdot \omega$

Table 1: Analogous terms in translational and rotational systems.

Part I
External Forces

2 The "Perfect" Turn

In racing, time is made and lost in turns. Cornering is critical and the path we drive through a turn has a huge effect on how fast we can get through a corner. This is because there is a natural limit to the cornering ability of a car: friction. If the tires cannot supply enough friction to hold the car to the turning radius, they will slip and control will suffer. In this section we will explore ways to use the geometry of the turn to get the most out of this frictional limit.

Consider the following: a particular 90° corner has a perfectly circular inner radius, $R1$, and outer radius, $R2$, as shown in Figure 2 (a). We assume that the car takes the turn at constant speed and follows a perfectly circular path (assumptions that we will scrutinize below). In this case, the car obeys the laws of circular motion and the centripetal force is the frictional force of the tires holding the car in a circular path. The maximum centripetal force that the tires can supply is given by

$$F_{friction} = \mu m g = m\frac{v_t^2}{r}.$$

The normal force is just the weight of the car, mg. However, mass cancels from the equations, so the weight of the car does not matter to this analysis. Also, we will assume a coefficient of static friction, μ, of 1.0. Solving for v_t gives the maximum speed at which the car can take a turn of radius r without losing traction:

$$v_t = \sqrt{gr}. \tag{1}$$

This makes sense: the larger the turning radius, the faster we can take the turn. The square root implies that if we quadruple the radius, for example, we should be able to take the turn at twice the speed. g is the gravitational acceleration ($9.8\,{}^{\mathrm{m}}/_{\mathrm{s}^2}$ or $32\,{}^{\mathrm{ft}}/_{\mathrm{s}^2}$), and is more or less constant so long as we are driving on Earth.

Returning to Figure 2 (a), two radii are already defined: $R1$ and $R2$. A turning radius of $R1$ would imply that we hug the inside of the turn all the way around, not the fastest path by any means. If we instead stay on the outside of the turn all the way around, our turning radius is $R2$. Because

$R2$ is greater than $R1$, Equation 1 says that this is the faster path - but not the *fastest*. As you may already know, a faster racing line is generally an "outside-inside-outside" path: starting from the outside edge of the track, touching the inside edge about halfway through the turn (the "apex") and finishing on the outside again. This racing line can be *approximated* as a circle that touches (is "tangent" to) the outside edges of the track and the halfway point of the inside edge of the turn, with a radius of $R3$ as drawn in Figure 2 (b).

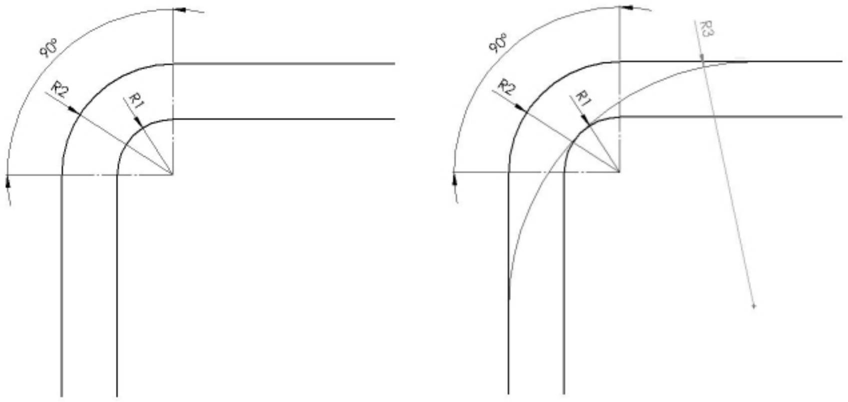

Figure 2: The 90° turn under consideration (a) and a large-radius racing line through it (b).

Our task: Find an equation so that if we know $R1$ and $R2$, we can solve for $R3$, the radius of the racing line. Once we know the radius, we can calculate what the increase in speed is from Equation 1. To help solve for $R3$, some construction lines are added to the drawing in Figure 3. Note the right triangle created: the construction lines have reduced the question to a simple trigonometric problem. (Convince yourself that the dimensions given on the hypotenuse and leg of the triangle are correct.)

Since we know the 45° angle, as well as the side adjacent to it and the hypotenuse, we can write the equation

$$cos(45°) = \left(\frac{R3 - R2}{R3 - R1}\right).$$

All that remains is to solve for $R3$,

$$R3 = \frac{R1cos(45°) - R2}{cos(45°) - 1}. \quad (2)$$

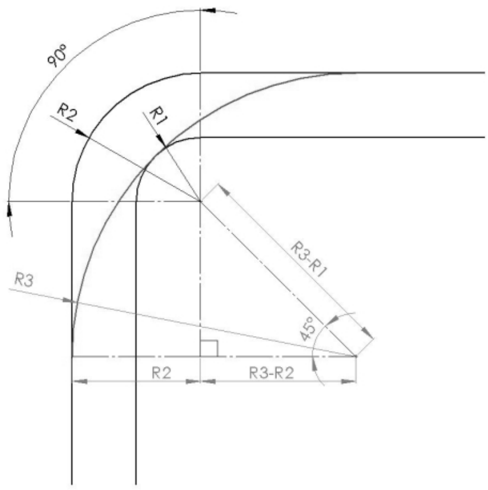

Figure 3: The turn with some construction lines added to help define the geometry.

This give R3 in terms of R1 and R2, which is what we were looking for. (Check for yourself that the algebra works out.) We will not prove it here, but this formula can be generalized to non-90° turns by replacing $cos(45°)$ with $cos(\frac{\theta}{2})$, where θ is the angle of the turn. (180° is an interesting case. Can you see why?)

As is the case with many physics problems, we have gone through a lot of geometric setup just to get to the interesting part. Now that we have a mathematical way of looking at a racing line, we can evaluate just how much faster it really is. To do so, let's take an example of a 90° turn with an inside radius, $R1$, of 150 feet and an outside radius, $R2$, of 250 feet. For reference, we can calculate the maximum speed we could take the turn by following the inner or outer radius using Equation 1:

$$v_{R1} = \sqrt{(g)(R1)} = \sqrt{(32^{\mathrm{ft}}/_{\mathrm{s}^2})(150\mathrm{ft})} = 69.3^{\mathrm{ft}}/_{\mathrm{s}} = 47.2^{\mathrm{mi}}/_{\mathrm{hr}}$$

$$v_{R2} = \sqrt{(g)(R2)} = \sqrt{(32^{\mathrm{ft}}/_{\mathrm{s}^2})(250\mathrm{ft})} = 89.4^{\mathrm{ft}}/_{\mathrm{s}} = 61.0^{\mathrm{mi}}/_{\mathrm{hr}}$$

As expected, the outside radius allows a higher speed through the turn. But now we calculate the radius of the racing line using Equation 2,

$$R3 = \frac{R1cos(45°) - R2}{sin(45°) - 1} = \frac{150\mathrm{ft}sin(45°) - 250\mathrm{ft}}{sin(45°) - 1} = 491.4\mathrm{ft},$$

and we find the maximum speed we can hold at that radius,

$$v_{R3} = \sqrt{(g)(R3)} = \sqrt{(32^{\mathrm{ft}}/_{\mathrm{s}^2})(491.4\mathrm{ft})} = 125.4^{\mathrm{ft}}/_{\mathrm{s}} = 85.5^{\mathrm{mi}}/_{\mathrm{hr}}.$$

There we have it: a gain of more than $24^{\mathrm{mi}}/_{\mathrm{hr}}$ through the turn!

Those of you who have been watching carefully may be tempted to call us liars: Yes, the speed through the turn increases as we consider wider and wider radii, but so does the distance. In fact, the speed only increases proportionally to $\sqrt{r}$, while the distance, a fraction of a circumference, increases proportionally to r ($C = 2\pi r$). So if we quadruple the radius of the turn, the distance we must travel also quadruples, but our speed only doubles. The time it takes to complete the turn, given by the distance divided by the speed, *doubles*! Is our analysis no good, then? On a track made up of all turns (like a big circle, or some NASCAR ovals), maybe. Then, it makes sense to just hug the inside line all the way around. But on a track with straightaways, the time lost braking for a turn and accelerating back up to speed after the turn are critical, more important than the time it takes to actually get through the turn. So in most cases it makes sense to preserve as much of the cars momentum as possible by finding the fastest line through a turn.

We stress that this analysis is an *approximation* only, and the real "perfect" turn depends on many other factors. The above paragraph is just one example. The assumption that we take the turn at constant speed is also unrealistic; braking and acceleration happen throughout. Consider the concept of the "late apex," well-known to race drivers, where it is better to brake later and longer coming into a turn, take a tighter radius at first, touch the inside of the turn after the midpoint, and accelerate out on a very wide path (see Figure 4). Besides the more qualitative reasons (better visibility coming out of the turn, less chance of running off the outside of the track), there is a good physical basis for this. Depending on the exact conditions, cars can usually apply nearly the maximum allowable braking force (before locking up the wheels) all the way into a turn, known as "threshold braking." Aerodynamic drag also helps slow the car down during the approach to a corner. Coming out of a turn, however, the car must fight aerodynamic drag and will usually reach its engine's power limit at some point. After that point, the tires will no longer be applying maximum frictional force. In other words, it generally takes less time for the car to slow down than to speed up. This asymmetry prompts the "late apex," which sacrifices some additional braking for a straighter, faster, and longer path on which to accelerate up to speed.

The right line through a turn also depends a lot on what comes before and after that turn. In a chicane or s-turns, exit speed from the final turn is most important (especially if it is followed by a long straightaway). It makes

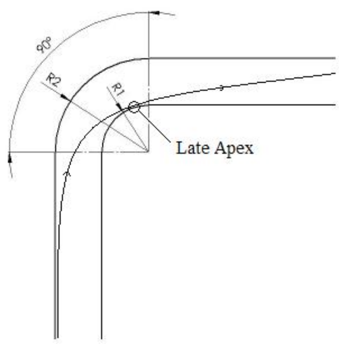

Figure 4: The “late apex” line.

sense, then, to take slower lines through the earlier turns in order to set up a fast line through the last one. In general, cornering is very much driven by knowledge of the track and the car, and is developed through practice. To model turning physically, many more factors than the ones we considered must be taken into account. (Gran Turismo does a fairly good job of modeling these factors in its physics engine, but it is not a perfect representation of real life.) Physics offers a lot of good approximations, and the analyis we covered can help us understand cornering, but it is important to realize that the real situation can often be more complex than our model.

3 Case Study: Car Suspension

Advanced Material: A non-trivial car design problem is the suspension system. An integral part of handling capabilities in an automobile, the suspension has remained nearly unchanged for almost one hundred years. Below is the suspension in a Honda Accord 2005 Coupe.

The goal of a suspension system is both control and comfort. The suspension should absorb the shock of the bumpy road. We can simplify our analysis somewhat by considering the pretty accurate suspension model shown in Figure 6. The image on the left is a graphical representation of the suspension. The mass is a quarter of the total mass of the car (since there are four wheels) connected to a spring and a damper (shock absorber), both of which have well-defined physical models and constitutive equations (we leave it to your physics textbook to explain these in detail). We also model the tire as a spring.

On the right is a diagram with some constants defined. Do not be fooled by the simplicity of the drawing. This problem is actually quite difficult!

Figure 5: Suspension for a Honda Accord 2005 Coupe.

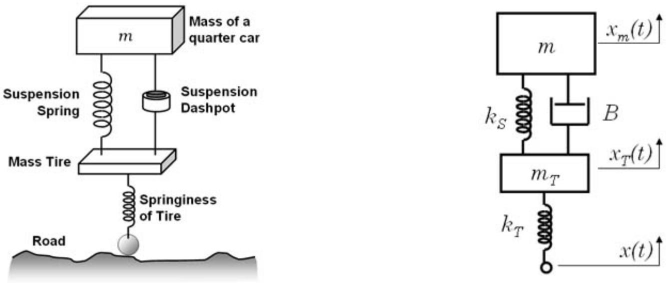

Figure 6: (a) Model for a suspension. (b) Diagram with variables defined.

Closed form solutions for the response of the car to an arbitrary input are impractical and entail messy differential equations. Instead, computer programs such as MatLAB are used.

The following plot is an output from a MatLAB script. It attempts to give a sense of how the suspension system reacts to an input (the large graph in the bottom) such as a speed bump. Notice how the response lasts longer than the input.

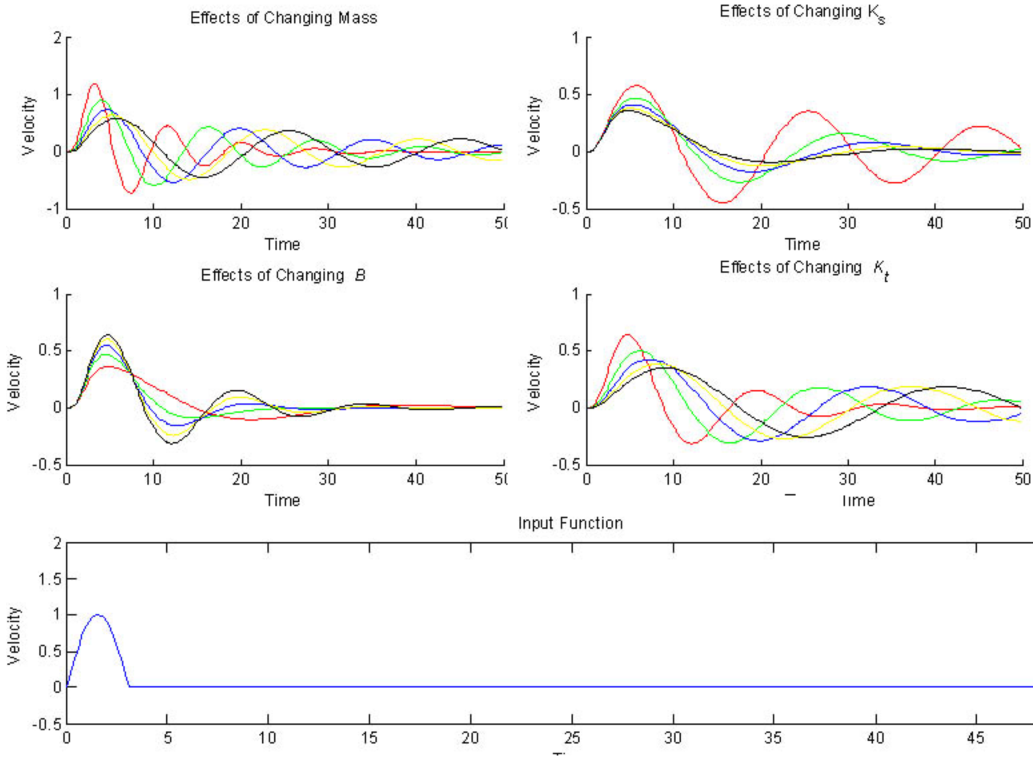

Figure 7: Effects of system constants on the suspension response. For each parameter the values plotted are 1 through 5, with red being 1.

Note for example the affect of changing the spring constant (top right graph). As the spring is made stiffer (black curve = softest, red curve = stiffest), more of the bumpiness in the road is transfered to the car, making for a rougher ride. However, stiffer springs might also be desireable because they reduce the amount of "body roll" a car experiences in a turn, making it feel more responsive. Racing teams attempt to find a balance between these characteristics of the car that works well on each track.

4 Can an F1 car drive on the ceiling?

This has perhaps become the trademark example of our Racing Physics course, and the closest we will get to a *Mythbusters* physics approach. The myth is simple: It has often been stated that an F1 car has enough downforce from its aerodynamic geometry that it could literally drive upside down on the ceiling at high enough speeds. By “ceiling,” we mean an inverted race track with all the same characteristic of a regular track except that gravity now pulls you away from the pavement. It certainly has never been attempted. Cars have done loops, but in these cases it is the centripetal normal force of the track that holds them in circular motion around the loop. What we are talking about is sustained driving on a flat surface. To determine if it is physically possible, we will have to draw from a lot of different concepts from both parts of this course. But don't worry: all of the formulas are simple enough to learn on the fly (no pun intended).

A good place to start would be to consider all the vertical forces on the car. Figure 8 (a) shows the scenario in question and Figure 8 (b) shows a partial free body diagram of the F1 car with only vertical forces.

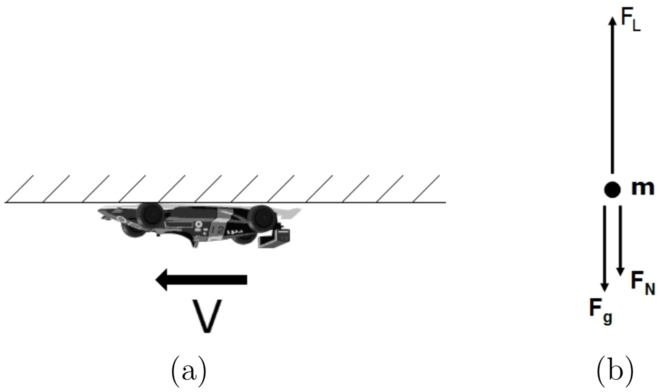

Figure 8: An F1 car in a precarious position (a) and a partial free body diagram showing only vertical forces (b).

The first of the vertical forces acting on the car is F_g, the force of gravity, which can be easily found if we know the mass of the car, $F_g = mg$. Gravity acts downward, as usual, but is now pulling the car *away* from the road instead of towards it. We also know that there is some aerodynamic force acting on the car. This force on a regular track is referred to as the “downfore” because it holds the car to the road for increased traction. The “downforce,” F_L, is now acting upwards since it is dependent on the orientation of the aerodynamic surfaces of the car, so it still acts to hold the car to the road. Finally, the road can exert some normal force, F_N. There are two conditions that can exist: If the net vertical force on the car (excluding the normal force) is downward, then the road doesn't have to exert any force on the car, since the car will promptly fall away from the upside down road and come crashing to the ground. If, though, the upwards aerodynamic force, F_L, is greater than the force of gravity, F_g, the road will exert whatever normal force is necessary to keep the car from passing through it. In other words, the vertical acceleration of the car will always be zero because it will remain at the height of the road. Therefore, since $F_{vertical} = ma_{vertial}$, the sum of the vertical forces must be zero as well.

Before we go any further, we will need the formula for calculating "downforce," and some specifications for the F1 car. Recall that the vertical aerodynamic force can be related to the so-called "coefficient of lift" via the equation

$$F_L = (C_L)\left(\frac{1}{2}\rho v^2 A\right).$$

The density of air, ρ, about $1.2^{\mathrm{kg}}/_{\mathrm{m}^3}$. (Can you see why this is in the equation? What would happen if we tried this on the moon?) v is the velocity of the car. The faster it goes, the more downforce we're going to get. A is the frontal area of the car, the area that the air has to go around as the car slices through it. We will use $A = 1.68\mathrm{m}^2$, a number typical of an F1 car[1]. C_L is the lift coefficient, which is determined by the geometry of the car, including its front and rear wings. We will use $C_L = 2.2$, also an estimate for F1 cars. We will also need to know the mass of the F1 car. For that, we will use $m = 600\mathrm{kg}$, the F1 speficication minimum allowable weight (including the driver). Now that we've got all that out of the way, let's calculate the vertical forces for an F1 car going 200 kilometers per hour (124MPH or $56^{\mathrm{m}}/_{\mathrm{s}}$):

$$F_g = mg = (600\mathrm{kg})\,(9.8^{\mathrm{m}}/_{\mathrm{s}^2}) = 5,880\mathrm{N},$$

$$F_L = (C_L)\left(\frac{1}{2}\rho v^2 A\right) = (2.2)\left(\frac{1}{2}\right)(1.2^{\mathrm{kg}}/_{\mathrm{m}^2})\,(56^{\mathrm{m}}/_{\mathrm{s}})^2\left(1.68\mathrm{m}^2\right) = 6,954\mathrm{N}.$$

The upwards-acting "downforce," F_L, wins out over gravity, meaning that even at this relatively low speed (for an F1 car), the car could stick to the ceiling at least momentarily. To have zero net vertical force, the road must exert a normal force, $F_N = F_L - F_g = 1,074\mathrm{N}$, pushing back down against the car. (If the road was not there, the car would momenetarily "fly" upwards!) This myth has passed its first test of feesibility.

As the skeptics will tell you, this, by itself, is not sufficient to prove that

[1] Specifications for an F1 car are not easy to find and are not usually published by racing teams. The data used in the following analysis comes from *Race Car Aerodynamics*, by Joseph Katz, Ph.D, ©2002 Robert Bently, Inc. An exerpt is available online at http://www.bentleypublishers.com/gallery.htm?code=GAER&galleryId=768.

an F1 car could drive continuously on the ceiling. If the car is just barely being held to the upside-down track, the tires will not have any traction and will not be able to fight the high drag force acting on the car at these speeds. The car will quickly slow down, losing its "downforce" and falling off the track. So, it seems that in order to test the critereon that the F1 car be able to drive on the ceiling continuously, we will need to go back to our free body diagram and include horizontal forces this time, as in Figure 9.

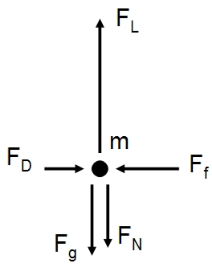

Figure 9: The F1 car's free body diagram, now with horizontal forces included.

In the free body diagram, the car is moving to the left. The two horizontal forces acting on the car are the drag force, F_D, trying to slow the car down, and the force that the tires exert to propel the car forward. Remember that this forward propulsion comes from the force of static friction between the tires and the pavement, so we call it F_f and note that it has a maximum value given by, $F_f^{max} = \mu F_N$, where μ is the static friction coefficient between the tires and the pavement. We will use $\mu = 1.0$, a low estimate for an F1 car. That means that our frictional force is simply equal to the normal force, F_N, of the road pushing down on the car, which we already calculated to be $1,074$N. To calculate the drag force, we use the same formula as for calculating the "downforce," except we replace the coefficient of lift, C_L, with the coefficient of drag, C_D, somewhere around 0.7 for an F1 car:

$$F_L = (C_D)\left(\frac{1}{2}\rho v^2 A\right) = (0.7)\left(\frac{1}{2}\right)(1.2^{\mathrm{kg}}/_{\mathrm{m}^2})\,(56^{\mathrm{m}}/_{\mathrm{s}})^2\left(1.68\mathrm{m}^2\right) = 2,213\mathrm{N}.$$

The drag force is larger than the maximum frictional force the tires can apply. The car will be slowed down by the air resistance until it falls off our upside-down track! So is the myth busted? Not quite: We can certainly squeeze more speed out of an F1 car. This will mean more downforce holding the car to the track. In turn, the track will push back with more normal force, allowing the tires more traction and more frictional force to propel the car forward. But, there will also be more drag force at higher velocity. Before we waste our time doing the math, lets consider if the benefits of more downforce will outweight the extra drag. The coefficient of lift, 2.2, is significantly higher than the coefficient of drag, 0.7. This means that for a given increase in velocity, we will get a larger increase in downforce than in drag by a factor of $\frac{2.2}{0.7} = 3.14$, referred to as the "lift to drag ratio" of the car. So going faster could potentially give us the edge we need to drive on the ceiling. Let's re-calculate all our forces with a new speed of 300 kilometers per hour (186MPH or $83\,{}^{\mathrm{m}}\!/\!{}_{\mathrm{s}}$):

$$F_L = (C_L)\left(\frac{1}{2}\rho v^2 A\right) = (2.2)\left(\frac{1}{2}\right)(1.2\,{}^{\mathrm{kg}}\!/\!{}_{\mathrm{m}^2})\,(83\,{}^{\mathrm{m}}\!/\!{}_{\mathrm{s}})^2\left(1.68\mathrm{m}^2\right) = 15,277\mathrm{N}.$$

The downforce nearly doubles. We are off to a good start. The weight of the car remains unchanged at $5,880$N. To balance out the vertical forces, the road must now push down with a normal force $F_N = F_L - F_g = 9,397$N. Since we assumed a static friction coefficient of 1.0, the maximum frictional force the tires can apply to propel the car forward is exactly the same as the normal force the road now exerts on the tires, $F_f = F_N = 9,397$N. Now the moment of truth: If the new drag force is less than the maximum frictional force the tires can supply, then the car will have enough traction to drive drive continuously upside-down:

$$F_D = (C_D)\left(\frac{1}{2}\rho v^2 A\right) = (0.7)\left(\frac{1}{2}\right)(1.2\,{}^{\mathrm{kg}}\!/\!{}_{\mathrm{m}^2})\,(83\,{}^{\mathrm{m}}\!/\!{}_{\mathrm{s}})^2\left(1.68\mathrm{m}^2\right) = 4,860\mathrm{N}.$$

The tires can lay down enough forward force to fight the drag force and more! To be sure, we should check to see that the engine itself can supply enough power to the wheels. (We suspect that it can, since 300 kilometers per hour is well within the reach of an F1 car.) To do this, we just have to know the power of the engine and the formula $Power = Force \cdot Velocity$. (We will officially meet power in Part II of the course.) An F1 engine can produce roughly 700 horsepower, or in SI units roughly 500,000 Watts of power (imagine 5,000 100-Watt lightbulbs!). So we calculate,

$$Force = \frac{Power}{Velocity} = \frac{500,000\mathrm{W}}{83\,{}^{\mathrm{m}}\!/\!{}_{\mathrm{s}}} = 6,024N.$$

This is less than the maximum frictional force the tires can supply, but still greater than the drag force. So we won't be able to peel out on the ceiling (probably not a good thing anyway), but the engine can definitely supply enough power to the tires to fight the drag force and keep the car moving forward on the upside-down track. We have effectively proven that an F1 car moving at 300 kilometers per hour can not only stick to the ceiling, but continuously drive on it so long as it maintains speed. The extreme aerodynamics of these cars make them exert much more than their own weight on the road, which is what gives them their unique handling abilities at extreme speeds.

Part II

Internal Forces

5 Power/Torque Curves

Two of the most important specifications of a race car are its power and torque. In the US, power is usually given in horsepower and roughly correlates to the car's overall ability to go fast (remember $P = F \cdot v$). Torque, given in foot-pounds (ft-lb) in the US, is more closely related to acceleration. (Torque measures the twisting force of the engine output shaft, which goes through the transmission and drive shaft to the wheel, causing them to exert a fricitonal force on the ground, which in turn makes the car accelerate.) It is important to note that when power or torque are cited as single number ("Car X has 200 horsepower and 150 foot-pounds of torque"), these are the *maximum* values for each. In actuality, power and torque vary depending on how fast the engine is turning.

The power/torque curve shows the values of power and torque as they vary with engine RPM, giving a more complete representation of an engine's output. They show the power and torque of the *engine*, before the transmission, so they are the same no matter what gear the car is in. Figure 10 is an example of a power/torque curve taken from Gran Turismo 4[2]. It is for a Toyota Altezza Touring Car (sold in consumer form as the Lexus IS in America). Notice that the scale for the torque curve (0 to 297 in horsepower) is different than the scale for the power curve (0 to 191 in ft-lb). Often, the two are plotted on the same scale for simplicity. We've imported the data (freehand) into a charting program and adjusted the scales to match. The result is Figure 11.

Notice that both torque and power are low at very low RPM. They increase with RPM until some maximum value, then begin to decrease at very high RPM. Torque peaks before power and in general is higher than power in the low RPM range. This becomes important in when considering gear changes, which we will look at below.

We won't go into the operation of an internal combustion engine, but we can use physics and a little bit of common sense to work out why the curves

[2]Gran Turismo 4 ©2005 Sony Computer Entertainment, Inc. All manufacturers, cars, names, brands and associated imagery featured in this game are trademarks and/or copyrighted materials of their respective owners.

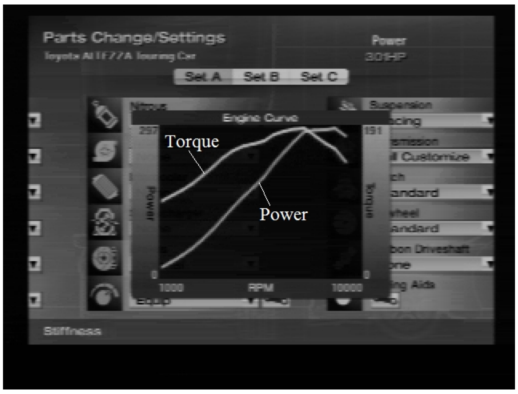

Figure 10: The power/torque curve of a Toyota Altezza Touring Car from Gran Turismo 4.

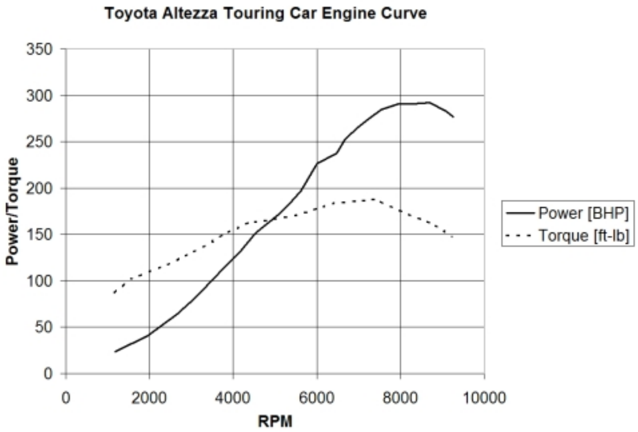

Figure 11: The same power torque curve, but adjusted so that the scales of each curve match.

look the way they do. As a starting point, consider an engine running at zero RPM (i.e. not running). It obviously produces no power and no torque, and so the two curves must start at a value of zero. As we increase the speed of the engine, we are injecting more fuel and combusting it more often. This increase in energy translates into a higher power output of the motor. The power peaks at a particular RPM where the engine is "happy," the timing of mechanical and fluid (combustion) events is as good as it can get. If we increase the RPM more, the speed of the pistons moving up and down in the engine is faster than the fuel combustion can optimally achieve, so some power is sacrificed. (Of course, if the engine gets going too fast, other problems like heat and vibration become limiting factors as well. This is the purpose of the "redline," to alert you that the engine is being stressed too much.)

One thing we haven't mentioned is how the torque and power curves are related to each other. The assumption has been that they are independent quantities that depend on engine characteristics. But remember from our defininitions of power that power and torque can be related by $P = \tau \cdot \omega$, where τ *is* the torque and ω is the angular velocity (speed of rotation) of the engine. So, in theory, if we know the torque and the speed of the engine, we should know its power. The only somewhat tricky part to this is that the angular velocity, ω, has to be in radians per time. (A radian is a unitless measure of angular rotation used to simplify many calculations. For us, it is only important to know that there are 2π radians in one full revolution.) As an example, let's try to find the power of a car that produces 150 ft-lb of torque at 5,000 RPM:

$$\begin{aligned} P &= \tau \cdot \omega \\ &= (150\text{ft} \cdot \text{lb})\,(5,000\,{}^{\text{rev}}\!/\!_{\text{min}})\,(2\pi\,{}^{\text{rad}}\!/\!_{\text{rev}}) \\ &= 4,712,389\,{}^{\text{ft}\cdot\text{lb}\cdot\text{rad}}\!/\!_{\text{min}}. \end{aligned}$$

To finish the calculation, we need to convert to horsepower. The conversion ratio is $1\text{hp} = 33,000\,{}^{\text{ft}\cdot\text{lb}}\!/\!_{\text{min}}$. We can ignore radians because it is a unitless measure (which is what makes it useful). So,

$$P = (4,712,389\,{}^{\text{ft}\cdot\text{lb}}\!/\!_{\text{min}}) \left(\frac{1}{33,000}\,{}^{\text{hp}}\!/\!_{\text{ft}\cdot\text{lb}/\text{min}} \right) = 143\text{hp}.$$

So that car, at that particular speed and torque, must be producing 143 horsepower. We can generalize this to say the torque and horsepower are always related in the same way, regardless of the car and the speed. What then of our power/torque curves? Figure 12 shows what happens if we apply the same calculation to every point on the Toyota Altezza's torque curve. As it happens, the power curve *is* almost the one we could calculate from the torque curve. There are some differences, which may be due to the fact that the two could have been measured independently in real life, but as far a physics is concerned, knowing one curve is enough to define fully the output of the motor.

Another interesting phenomenon that can now be explained is the fact that all power/torque curves where both are plotted on the same scale intersect at around 5,250 RPM. If we ignore units (which we are implicitly doing

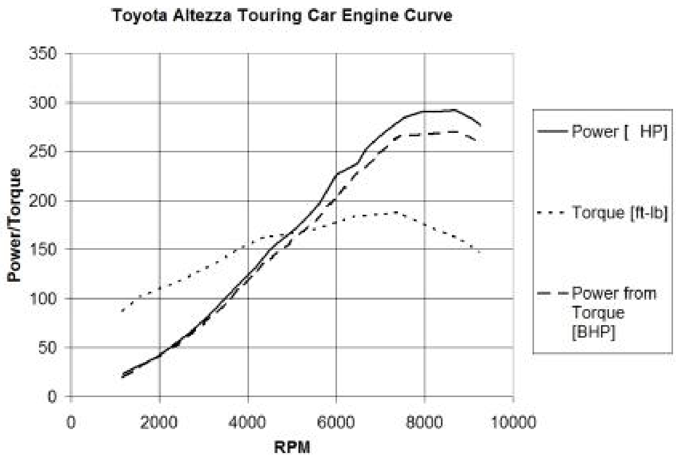

Figure 12: The Altezza power/torque curve, with the calculated power from the torque curve shown as well.

by graphing both curves on the same scale), then we can see that this comes from the same calculation as we carried out above:

$$HP = (Torque)(RPM)(2\pi)(\frac{1}{33,000}).$$

Horsepower will exactly equal torque when RPM cancels out the conversion factors. To do this, it will have to be equal to their reciprocal. That is,

$$RPM = \frac{33,000}{2\pi} = 5,252.$$

The Altezza curves (when graphed on the same scale in Figure 12) interesect at approximately this number. If we had done the analysis with any other car's curves, the intersection point would be the same.

6 Gearing

As we saw in the previous section, power and torque have a maximum value in a particular range of RPM. This is often referred to as a car's "power band," and it varies depending on the car. Since the power band is fairly narrow for an internal combustion engine, using it effectively requires that we have multiple gear ratios to choose from for transmitting the power to the road, hence, shifting. Before we look at how shifting helps us stay inside a car's power band, let's take a quick look at the physics of gears:

If a car engine's output shaft were hooked directly to the wheels, it would provide very little force for pushing the car forward. The Altezza Touring Car's engine outputs a maximum of just under 200 ft-lb of torque. If that torque were acting directly on the wheels of the car, the force provided can be calculated simply (remember $\tau = r \times F$). The radius, r, is that of the wheel and tire. If the radius is 14 inches, we can plug in and solve for the force:

$$\tau = r \times F \Rightarrow F = \frac{\tau}{r} = \frac{200\text{ft} \cdot \text{lb}}{(14\text{in})\left(\frac{1}{12}{}^{\text{ft}}/_{\text{in}}\right)} = 171\text{lb}.$$

A (strong) human being could push the car with that amount of force. And if the car were going up even a slight incline, the force of gravity would cancel out this force. All is not lost, though, because we have another component of power to draw from: the engine's speed. If it were directly driving the car's wheel at 7,500 RPM (the point of maximum torque), how fast would the car be going? We can calculate this easily by multiplying by the circumference of the wheel (this is how far the car travels in one revolution) and then converting to a sensible unit:

$$V = (7,500{}^{\text{rev}}/_{\text{min}})\,(2\pi \cdot 14{}^{\text{in}}/_{\text{rev}})\left(\frac{1}{63,360}{}^{\text{mi}}/_{\text{in}}\right)(60{}^{\text{min}}/_{\text{hr}}) = 625{}^{\text{mi}}/_{\text{hr}}!$$

From practical experience, we know that we will never reach these speeds. The goal, then, is to sacrifice the engine's high speed for more torque. This is exactly what the transmission does through the process of gear reduction. There are two stages of reduction in a typical car: the transmission gears and the final drive gear. The transmission gears are the ones that the driver can select (or, in an automatic, the car's computer chooses). The final drive gear is a characteristic of the car's differential, the module which splits the engine's power between the left and right wheels. Figure 13 is a simplified model of a car's gear reduction stages.

Without going into gear theory too much, we will just mention going from a smaller gear to a larger gear (one with fewer teeth to one with more teeth) translates to an increase in torque and a decrease in speed. To understand this, think about the geometry of the gears: the smaller gear must turn many times for the larger gear to turn once. However, the bigger gear has a larger radius, and thus the torque ($\tau = r \times F$) is also larger. The "gear ratio" refers

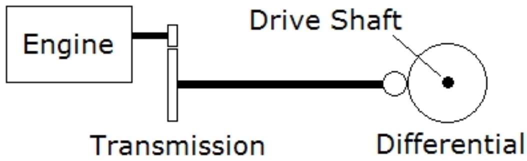

Figure 13: The power transmission from engine to drive shaft involves two gear stages, one at the transmission itself and one at the differential.

to the exact amount of torque increase and speed decrease we get. For example, a 30-tooth gear meshing with a 90-tooth gear of the same pitch (size and spacing of teeth) gives a gear ratio of $\frac{90}{30} = 3$. The ratio is always given with the output gear on the top of the fraction and the input gear on the bottom. The torque increase in this case would also be a factor of 3: $\tau_{out} = 3\tau_{in}$. The speed changes by the opposte: a factor of $\frac{1}{3}$, so that $\omega_{out} = \frac{1}{3}\omega_{in}$. Note that no matter what the gear ratio is, power is conserved. ($P = \tau \cdot \omega$, the increase in torque exactly cancels the decrease in speed, $3\tau \cdot \frac{1}{3}\omega = \tau \cdot \omega$.)

Gear ratios can be "chained" together simply by multiplying them. For example, if our transmission gear ratio is 3:1 and our final drive gear ratio is 4:1, the total gear ratio from the engine to the drive shaft is 12:1 (this can also be written as $\frac{12}{1}$ or simply 12). The torque increases by a factor of 12 and the speed decreases accordingly to $\frac{1}{12}$ the engine speed.

We haven't yet looked at the effect of having multiple gears to choose from. Let's return to the Toyota Altezza Touring Car and look at its trasmission settings (an important option for tweaking in Gran Turismo 4). Figure 14 shows the table of gears, from 1st to 5th, as well as the final drive gear.

Given that this car's engine supplies maximum power at around 8,500 RPM, let's work out the straight-line speed at which we hit this maximum power for each gear ratio. First, we divide the engine RPM by the total gear reduction to get the RPM of the drive shaft. Then, we multiply by the wheel circumference and convert to miles per hour as we did above:

$$V_1 = \frac{8,500^{\mathrm{rev}}/_{\mathrm{min}}}{(3.890)\,(4.100)}\,(2\pi \cdot 14^{\mathrm{in}}/_{\mathrm{rev}})\left(\frac{1}{63,360}{}^{\mathrm{mi}}/_{\mathrm{in}}\right)(60^{\mathrm{min}}/_{\mathrm{hr}}) = 44^{\mathrm{mi}}/_{\mathrm{hr}}$$

Figure 14: The Altezza Touring Car's gear ratio settings.

$$
\begin{aligned}
V_2 &= \frac{8,500^{\mathrm{rev}}/_{\mathrm{min}}}{(2.504)\,(4.100)}\left(2\pi \cdot 14^{\mathrm{in}}/_{\mathrm{rev}}\right)\left(\frac{1}{63,360}{}^{\mathrm{mi}}/_{\mathrm{in}}\right)\left(60^{\mathrm{min}}/_{\mathrm{hr}}\right) = 69^{\mathrm{mi}}/_{\mathrm{hr}} \\
V_3 &= \frac{8,500^{\mathrm{rev}}/_{\mathrm{min}}}{(1.766)\,(4.100)}\left(2\pi \cdot 14^{\mathrm{in}}/_{\mathrm{rev}}\right)\left(\frac{1}{63,360}{}^{\mathrm{mi}}/_{\mathrm{in}}\right)\left(60^{\mathrm{min}}/_{\mathrm{hr}}\right) = 98^{\mathrm{mi}}/_{\mathrm{hr}} \\
V_4 &= \frac{8,500^{\mathrm{rev}}/_{\mathrm{min}}}{(1.320)\,(4.100)}\left(2\pi \cdot 14^{\mathrm{in}}/_{\mathrm{rev}}\right)\left(\frac{1}{63,360}{}^{\mathrm{mi}}/_{\mathrm{in}}\right)\left(60^{\mathrm{min}}/_{\mathrm{hr}}\right) = 130^{\mathrm{mi}}/_{\mathrm{hr}} \\
V_5 &= \frac{8,500^{\mathrm{rev}}/_{\mathrm{min}}}{(1.046)\,(4.100)}\left(2\pi \cdot 14^{\mathrm{in}}/_{\mathrm{rev}}\right)\left(\frac{1}{63,360}{}^{\mathrm{mi}}/_{\mathrm{in}}\right)\left(60^{\mathrm{min}}/_{\mathrm{hr}}\right) = 165^{\mathrm{mi}}/_{\mathrm{hr}}.
\end{aligned}
$$

For each successive gear, the speed at which we hit the maximum point on the power curve gets higher and higher. For slower speeds, we get more power out of the engine in lower gears, which can translate to faster acceleration. For faster speeds, the higher gears keep us in the power band of the car. From this, we can begin to lay out a shifting profile for the car. From a stop to somewhere above 44 mph, we should be in first gear. Between 44 and 69 mph, we shift into second gear, dropping the engine to a lower RPM so that we can go through the high part of the power curve again. We can do this in between each peak power speed to always keep the engine in the narrow range of RPM around the point of maximum power. Figure 15 shows this graphically.

From a stop, we would go through the power curve once. The last point on the curve is the engine redline, so we need to shift into second gear at this point. The shift occurs at the first vertical line, somewhere near 45 mph. We

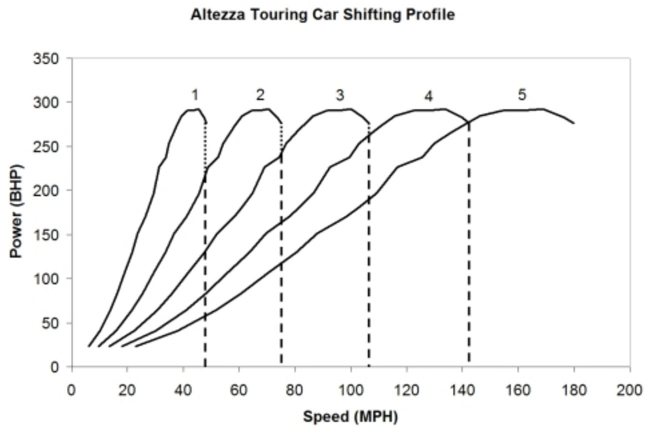

Figure 15: Scaling the power curve for each gear ratio gives a visual representation of what shifting does. Vertial lines indicate a shift point.

then follow the second gear power curve through the maximum point again. We redline and shift up again at about 75 mph into 3rd gear. By doing this, the engine is always outputting near max power (between 225 and 300 horsepower for the Altezza). This shifting profile varies from car to car depending on the shape of the power curve, the gear ratios, and the redline.

Racing teams will modify their gear ratios for each track. Short tracks with many corners require more acceleration (torque) than speed, so the gear reduction for all gears will be higher. Imagine the shifting profile of Figure 15 being squished to the left. Shifts would occur at lower speeds for each gear and the time spent in any particular gear would be less (more frequent shifting). The higher gear ratios lead to an overall increase in torque (and acceleration), but at the cost of speed (because power must be conserved). For a long, fast track with lots of straightaways, smaller gear reductions would be used to get more top speed out of the car. The shifting profile would be stretched to the right, making shift speeds for each gear higher and leading to less frequent shifting. The top speed of the car would increase, but the peak torque and acceleration would decrease.

Tradeoffs like this are very common in physics, where conservation of energy is a fundamental consideration. Balancing torque and speed by tuning the transmission ratios for a particular track is one of the most important ways to increase lap tme.

Advanced Material: Power and torque curves are one good way of defining a car's performance, but often other statistics like top speed and 0-60 or 0-100 acceleration are given. We have vaguely stated that a higher gear ratio leads to more torque and more acceleration, while a lower gear ratio leads to more top speed. To give a bit more shape to these relationships, we can combine a few of our basic equations,

$$P = F \cdot v, F = ma,$$

to give a relationship that involves both velocity *and* acceleration:

$$F = \frac{P}{v} = ma.$$

We also know that acceleration is the rate of change of velocity with respect to time, $a = \frac{\Delta v}{\Delta t}$. Acceleration is an *instantaneous* rate of change, though, and has a different value at every point in time. The tool that we use to solve equations like this is calculus, and the so-called "differential equation" of interest is

$$\frac{P}{v} = m\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = \frac{P}{mv}.$$

We can apply the methods of calculus to our gearing profile, which gives us power at a given speed, so that we have the entire right hand side of the differential equation (assuming we know the car's mass). Solving numerically (i.e. with a computer) will result in a plot of the speed of the car at any given time. With some modification to our computer program, we can also add in effects like the traciton limit of the tires, the drag force, and the delay involved when changing gears. The result will look something like Figure 16, a rough computation for the Altezza Touring Car.

A ton of information can be picked up from this graph, including the 0-60 time (a bit over 5 seconds), the 0-100 time (around 12 seconds), the top speed (61$^{\mathrm{m}}/_{\mathrm{s}}$ or 136MPH, and the time to reach top speed (35-40 seconds). Once the sequence of calculations that generates this graph is correct, it can be repeated to test different gear ratios. More interestingly, if the equations are good enough, the car's properties (mass, coefficient of drag, etc.) accurate enough, and the processor fast enough (to mimick "instantaneous" values), you can run these and other calculations over and over again to produce an accurate simulation of driving a race car. This is exactly how the Gran Turismo 4 physics engine does it.

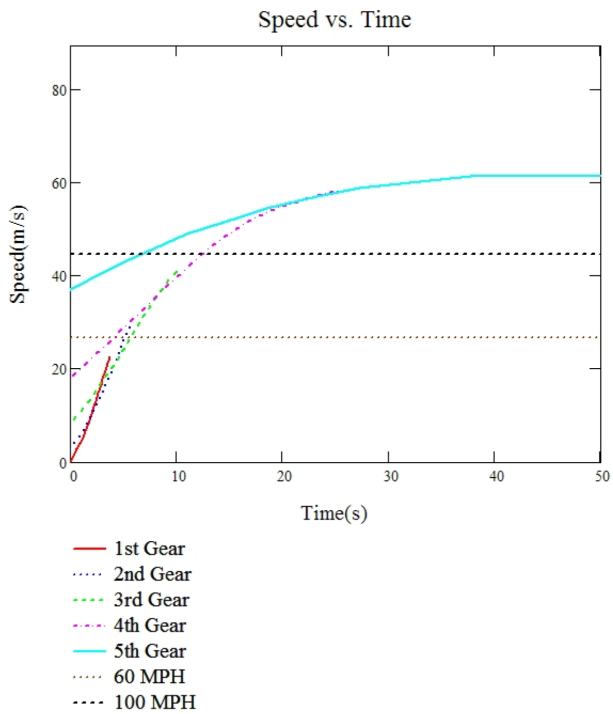

Figure 16: A computed speed vs. time graph for the Altezza Touring car with particular trasmission settings, accounting for estimated drag, traction limits, and shift lag.

7 The Tesla Roadster

This being an MIT course, we could not pass up the opportunity to look at one of the possible paths racing might take in the future. The world oil situation is such that hybrid electric vehicles have become a viable consumer alternative. Hydrogen fuel cell-powered cars seem primed to make a similar transition from prototype to consumer product, but the infrastructure will take longer to develop. Another alternative that hasn't been in the media much is the fully-electric "plug in" vehicle. Generally, alternative-power vehicles have been tagged as small and underpowered, and particularly in the case of the fully-electric vehicle, impractical due to long charging times and short range. Whereas environmentally-aware consumers may still buy them, these limitations have kept alternative-power vehicles out of the performance market altogether.

There is no inherent reason, however, why electric motors are inferior to internal combustion engines. In fact, electric motors offer benefits that make them in some ways more well-suited to racing than the internal combustion engine. When the low power "green vehicle" design stereotype is lifted away and an automobile is allowed to pursue the full potential of the electic motor, the result can be very exciting. Enter the Tesla Roadster.

Tesla Motors is a Silicon Valley startup (part of their venture capital came from Google and PayPal cofounders) devoted to making a performance electric vehicle available to the public [3]. Having such a vehicle could help produce more interest in electic cars and spark more development on the level of an every-day vehicle. Their first model, the Tesla Roadster, will be delivered in 2007 (not prototypes - actual consumer vehicles). The price tag is a hefty $100,000, but the performance does live up to the price.

The power plant of the Tesla Roadster is a 3-phase AC induction motor. AC stands for Alternating Current and is the type of electricity that comes in through a wall socket (although this is single phase). (Nikola Tesla was the father of AC power distribution and invented the AC induction motor.) Induction refers to the method by which torque is generated: Current passing through coils of copper wire in the motor create a magnetic field, which in turn induces currents in more conductors on the rotor, the actual rotating component of the motor. The currents are controlled so that the magnetic field rotates, which then causes the rotor to rotate as well[4].

We won't go any further into the theory of electric motors (which can get pretty complex) except to mention some of the characteristics that set them apart from internal combustion engines. For one, they are a lot simpler mechanically. There is one moving part: the rotor. No pistons, valves, cams, or timing belts are required and for that reason they can also be a lot more efficient, reliable and durable (assuming the electrical system is made well). They also have an extremely high power density (power generation compared to their weight). The Tesla Roadster's 248 peak horsepower motor weighs just 70 lbs. Scaling up to higher power would be relatively easy, but as we will soon see, a relatively low horsepower electric motor could outperform a higher horsepower internal combustion engine anyway.

[3] *Telsa Motors*. Wikipedia, 2006. http://en.wikipedia.org/wiki/Tesla_Motors.

[4] For more information on Tesla's inventions, see the Wikipedia article at http://en.wikipedia.org/wiki/Nikola_Tesla. For information about electric motors, see http://en.wikipedia.org/wiki/Electric_motor and http://www.coolmagnetman.com/magacmot.htm.

The key to the electric motor's performance lies in its torque curve. Figure 17 shows the power and torque curve of the Tesla Roadster[5]. Whereas an internal combustion engine delivers no torque at zero RPM, an electric motor can and does. Try stopping a small electric motor, like one from a toy car, with your hands. You'll notice that even when the shaft is not rotating, the motor still exerts a torque, trying to spin in your grip. In fact, electric motors generally supply their maximum torque "at stall" (zero RPM). In the Tesla Roadster, the torque stays high through a huge RPM range, past 8,000 RPM, and then begins to drop off more steeply. The engine redlines at 13,500 RPM, significantly higher than most internal combustion engines. (The small number of moving pieces help it to reach these high speeds without tearing itself apart like an internal combustion engine might.)

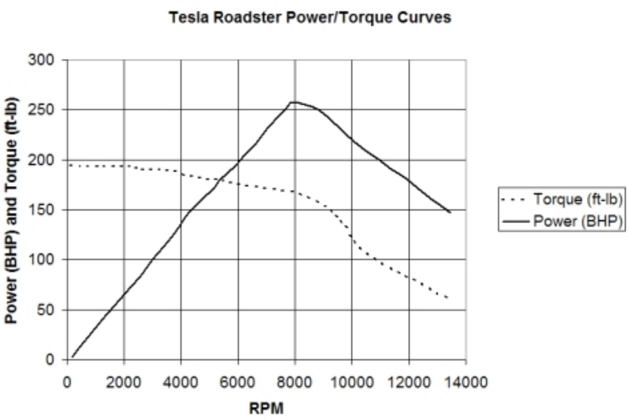

Figure 17: The Tesla Roadster power/torque curves. Compare this with the Altezza Touring Car curves in Figure 11. Notice the nearly constant torque throughout a large range of RPM. Notice also that the intersection point is still around 5,252 RPM, as we proved theoretically above.

What this torque curve translates to in terms of performance is blisteringly fast acceleration, even from a dead stop. (Remember, $\tau = r \times F$ and $F = ma$, so higher torque leads to higher force at the wheels, which in turn leads to higher acceleration.) The Tesla Roadster claims a 0-60 time of 4 seconds, on par with supercars like the Lamborghini Murcielago. (It should be noted that the Tesla Roadster isn't unique in its ability to produce quick acceleration with an electric motor. The GM EV1, featured in the movie

[5] These *approximate* power/torque curves are based on the figure on the Telsa Motors performance page, http://www.teslamotors.com/performance/performance.php. They are freehand approximations only and do not represent official data from Telsa Motors.

Who Killed the Electric Car? was often criticized for lack of power although it could do 0-60 in about 8 seconds, comparable to a 6-cylinder Ford Taurus[6].)

Because of the Tesla Roadster's wide power band and high redline, it needs fewer gears to stay within its motor's optimum RPM range. It has only two, with gear ratios of 4.20:1 and 2.17:1, plus a final drive ratio of 3.41:1. Figure 18 shows the Telsa Roadster power curve, scaled for each gear. The top speed of the Tesla Roadster is listed as over 130 mph. Top speed is more dependent on power than on torque (remember $P = F \cdot v$) and 250 hp is nothing special in the performance world. But the Tesla Roadster can get up to its top speed very quickly because of its low-RPM torque, and having fewer gears means less time lost during shifting. The Tesla Roadster would be particularly well suited, then, to a track with many turns and relatively short straightaways, where acceleration is more important than top speed.

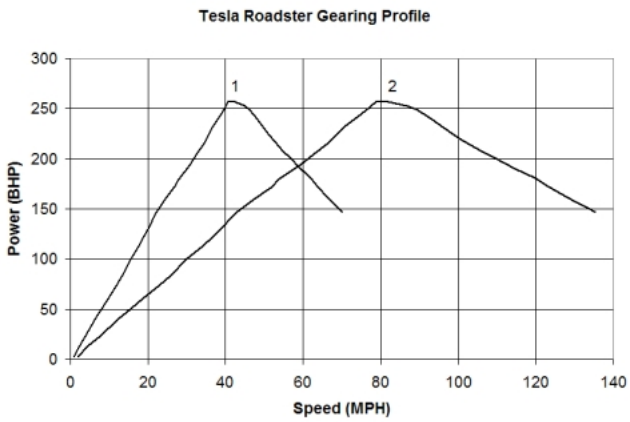

Figure 18: The gearing profile of the Tesla Roaster. Because of its wide power band, it can get up to a top speed of over 130 mph with only two gears. Compare this to the Altezza Touring Car's gearing profile in Figure 15.

[6] *General Motors EV1.* Wikipedia, 2006. http://en.wikipedia.org/wiki/General_Motors_EV1.

8 Concluding Remarks

Hopefully, this class has shown you that physics can and is applied in racing. While our methods were sometimes a bit complicated, they were all based on some fundamental equations of physics that come straight from your high school physics courses. Complicated situations can be broken down into simpler components, both in the quantitative sense (such as forces being resolved into vertical and horizontal components) and in the more figurative sense (seeing what contributes to a physical situation, what can vary, what can be ignored). Whether you go on to study physics as a pure science or in the applied form of engineering, the ability to look at a complicated situation with a “toolbox” of simpler building blocks will be essential. We hope you’ve enjoyed this course and will pursue other intersting and enjoyable topics in science and engineering. Feel free to contact us with questions about the course material or about any other questions you may have.

Shane and Dayán